

The Shannon capacity of a graph and the independence numbers of its powers

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Abstract

The independence numbers of powers of graphs have been long studied, under several definitions of graph products, and in particular, under the strong graph product. We show that the series of independence numbers in strong powers of a fixed graph can exhibit a complex structure, implying that the Shannon Capacity of a graph cannot be approximated (up to a sub-polynomial factor of the number of vertices) by any arbitrarily large, yet fixed, prefix of the series. This is true even if this prefix shows a significant increase of the independence number at a given power, after which it stabilizes for a while.

1 Introduction

Given two graphs, G_1 and G_2 , their *strong graph product* $G_1 \cdot G_2$ has a vertex set $V(G_1) \times V(G_2)$, and two distinct vertices (v_1, v_2) and (u_1, u_2) are connected iff they are adjacent or equal in each coordinate (i.e., for $i \in \{1, 2\}$, either $v_i = u_i$ or $v_i u_i \in E(G_i)$). This product is associative and commutative, and we can thus define G^k as the product of k copies of G . In [11], Shannon introduced the parameter $c(G)$, the *Shannon Capacity* of a graph G , which is the limit $\lim_{k \rightarrow \infty} \sqrt[k]{\alpha(G^k)}$, where $\alpha(G^k)$ is the independence number of G^k (it is easy to see that this limit exists by super-multiplicativity). The considerable amount of interest that $c(G)$ has received (see, e.g., [1], [2], [5], [6], [7], [8], [9], [10], [12]) is motivated by Information Theory concerns: this parameter represents the effective size of an alphabet, in a communication model where the graph G represents the channel. In other words, we consider a transmission scheme where the input is a set of single letters

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$V = \{1, \dots, n\}$, and our graph G has V as its set of vertices, and an edge between each pair of letters, iff they are confusable in transmission (i.e., $(1, 2) \in E(G)$ indicates that sending an input of 1 or an input of 2 might result in the same output). Clearly $\alpha(G)$ is the maximum size of a set of single letters which can be predefined, then sent with zero-error. By definition, $\alpha(G^k)$ represents such a set of words of length k (since two distinct words are distinguishable iff at least one of their coordinates is distinguishable), leading to the intuitive interpretation of $c(G)$ as the effective size of the alphabet of the channel (extending the word length to infinity, while normalizing it in each step).

Consider the series $a_k = a_k(G) = \sqrt[k]{\alpha(G^k)}$, which we call "the *independence series* of G ". As observed in [11], the limit $c(G) = \lim_{k \rightarrow \infty} a_k$ exists and equals its supremum, and $a_{mk} \geq a_k$ for all integers m, k . Our motivation for the study of the series a_k is the computational problem of approximating $c(G)$. So far, all graphs whose Shannon capacity is known, attain the capacity either at a_1 (the independence number, e.g., perfect graphs), a_2 (e.g., self complementary vertex-transitive graphs) or do not attain it at any a_k (e.g., the cycle C_5 with the addition of an isolated vertex). One might suspect that once the a_k series remains roughly a constant for several consecutive values of k , its value becomes a good approximation to its limit, $c(G)$. This, however, is false. Moreover, it remains false even when restricting ourselves to cases where a_k increases significantly before it stabilizes for a few steps. We thus address the following questions:

1. Is it true that for every arbitrarily large integer k , there is a $\delta = \delta(k) > 0$ and a graph G on n vertices such that the values $\{a_i\}_{i < k}$ are all at least n^δ -far from $c(G)$?
2. Can the series a_k increase significantly (in terms of $n = |V(G)|$) in an arbitrary number of places?

In this short paper we show that the answer to both questions above is positive. The first question is settled by Theorem 1, proved in section 2.

Theorem 1. *For every fixed $\nu \in \mathbb{N}$ and $\varepsilon > 0$ there exists a graph G on N vertices such that for all $k < \nu$, $a_k \leq c_k \log_2(N)$ (where $c_k = c_{k,\nu}$), and yet $a_\nu \geq N^{\frac{1}{\nu} - \varepsilon}$.*

Indeed, for any fixed k , there exists a graph G on N vertices, whose Shannon Capacity satisfies $c(G) > N^\delta \max_{i < k} \{a_i\}$, where $\delta = \frac{1 - o(1)}{k}$.

Theorem 2, proved in section 3, settles the second question, and implies the existence of a graph G whose independence series a_k contains an arbitrary number of "jumps" at arbitrarily chosen locations; hence, noticing a significant increase in this series, or noticing that it stabilizes for a while, does not ensure any proximity to $c(G)$.

Theorem 2. *For every fixed $\nu_1 < \dots < \nu_s \in \mathbb{N}$ and $\varepsilon > 0$ there exists a graph G such that for all $k < \nu_i$, $a_k < a_{\nu_i}^\varepsilon$ ($i \in \{1, \dots, s\}$).*

The above theorems imply that the naive approach of computing the a_k values for some k does not provide even a PSPACE algorithm for approximating $c(G)$. Additional remarks on the complexity of the problem of estimating $c(G)$, as well as several open problems, appear in the final section 4.

2 The capacity and the initial a_k -s

In this section we prove Theorem 1, using a probabilistic approach, which is based on the method of [3], but requires some additional ideas.

Let $2 \leq \nu \in \mathbb{N}$; define $N = n\nu$ (n will be a sufficiently large integer), and let $V(G) = \{0, \dots, N-1\}$. Let \mathcal{R} denote the equivalence relation on the set of unordered pairs of *distinct* vertices, in which (x, y) is identical to (y, x) and is equivalent to $(x + kn, y + kn)$ for all $0 \leq k \leq \nu - 1$, where addition is reduced modulo N . Let $\{\mathcal{R}_1, \dots, \mathcal{R}_M\}$ denote the different equivalence classes of \mathcal{R} . For every $x \neq y$, let $\mathcal{R}(x, y)$ denote the equivalence class of (x, y) under \mathcal{R} ; then either $|\mathcal{R}(x, y)|$ is precisely ν , or the following equality holds for some $l < \nu$:

$$(x, y) \equiv (y + ln, x + ln) \pmod{N}$$

This implies that $N \mid 2ln$, hence $2l = \nu$. We deduce that if ν is odd, $|\mathcal{R}_i| = \nu$ for all $1 \leq i \leq M$, and $M = \frac{1}{\nu} \binom{N}{2}$. If ν is even:

$$|\mathcal{R}(x, y)| = \begin{cases} \frac{1}{2}\nu & \text{If } y \equiv x + \frac{1}{2}\nu n, \\ \nu & \text{Otherwise.} \end{cases}$$

and $M = \frac{1}{\nu} \binom{N}{2} + \frac{1}{2\nu} N$, i.e., in case of an even ν there are $N/2$ pairs which belong to n smaller classes, each of which is of size $\frac{1}{2}\nu$, while the remaining edges belong to ordinary edge classes of size ν .

The edges of G are chosen randomly, by starting with the complete graph and excluding a single edge from each equivalence class, uniformly and independently, thus $|E(G)| = \binom{N}{2} - M = \binom{N}{2} \left(\frac{\nu-1}{\nu} + o(1) \right)$.

A standard first moment consideration (c.f., e.g., [4]) shows that $a_1 = \alpha(G) < \lceil 2 \log_\nu(N) \rceil$ almost surely. To see this, set $s = \lceil 2 \log_\nu(N) \rceil$, and take an arbitrary set $S \subset V(G)$ of size s . If S contains more than one member of some edge class \mathcal{R}_i , it cannot be independent. Otherwise, its edge probabilities are independent, and all that is left is examining the lengths of the corresponding edge classes. Assume S contains r pairs which belong to short edge classes: $(x_1, y_1), \dots, (x_r, y_r)$. If ν is odd, $r = 0$, otherwise $y_i = x_i + \frac{1}{2}\nu n$ for all i , and $x_i \not\equiv x_j \pmod{\frac{1}{2}\nu n}$ for all $i \neq j$ (distinct pairs in S belong to distinct edge classes). It follows that $r \leq \frac{s}{2}$, and we deduce that for each such set S :

$$\Pr[S \text{ is independent}] \leq \left(\frac{1}{\nu} \right)^{\binom{s}{2} - r} \left(\frac{2}{\nu} \right)^r \leq \left(\frac{1}{\nu} \right)^{\binom{s}{2}} 2^{s/2}$$

Applying a union bound and using the fact that $\frac{(2\nu)^{s/2}}{s!}$ tends to 0 as N , and hence s , tend to infinity, we obtain:

$$\Pr[\alpha(G) \geq s] \leq \binom{N}{s} \nu^{-\binom{s}{2}} 2^{s/2} \leq \frac{2^{s/2}}{s!} \left(N \nu^{-\frac{s-1}{2}} \right)^s = \frac{(2\nu)^{s/2}}{s!} \left(N \nu^{-\frac{s}{2}} \right)^s \leq \frac{(2\nu)^{s/2}}{s!} = o(1),$$

where the $o(1)$ term here, and in what follows, tends to 0 as N tends to infinity.

We next deal with G^k for $2 \leq k < \nu$. Fix a set $S \subset V(G^k)$ of size $s = \lceil c_k \log_2^k(N) \rceil$, where c_k will be determined later. Define S' , a subset of S , in the following manner: start with $S' = \emptyset$, order the vertices of S arbitrarily, and then process them one by one according to that order. When processing a vertex $v = (v_1, \dots, v_k) \in S$, we add it to S' , and remove from S all of the following vertices which contain $v_i + tn \pmod{N}$ in any of their coordinates, for any $i \in [k]$ and $t \in \{0, \dots, \nu - 1\}$. In other words, once we add v to S' , we make sure that its coordinates modulo n will not appear anywhere else in S' . If S is independent, it has at most $\alpha(G^{k-1})$ vertices with a fixed coordinate, thus $s' = |S'| \geq s / (k^2 \cdot \nu \cdot \alpha(G^{k-1}))$. Notice that each two distinct vertices $u, v \in S'$ have distinct vertices of G in every coordinate, thus $\mathcal{R}(u_i, v_i)$ is defined for all i ; furthermore, for any *other* pair of distinct vertices $u', v' \in S'$, the sets $\{\mathcal{R}(u_1, v_1), \dots, \mathcal{R}(u_k, v_k)\}$ and $\{\mathcal{R}(u'_1, v'_1), \dots, \mathcal{R}(u'_k, v'_k)\}$ are disjoint.

We next bound the probability of an edge between a pair of vertices $u \neq v \in S'$. Let k' denote the number of **distinct** pairs of corresponding coordinates of u, v , and let t_l , $1 \leq l \leq M$, be the number of all such distinct pairs whose edge class is \mathcal{R}_l (obviously $\sum_{l=1}^M t_l = k'$). For example, when all the corresponding pairs are distinct, we get $k' = k$ and $t_l = |\{1 \leq i \leq k : \mathcal{R}(u_i, v_i) = \mathcal{R}_l\}|$. Notice that, by definition of S' , for every i , $v_i \neq u_i + \frac{1}{2}\nu n$, and thus $\mathcal{R}(u_i, v_i)$ is an ordinary edge class. It follows that:

$$\Pr[uv \in E(G^k)] = \prod_{l=1}^M \frac{\nu - t_l}{\nu} \quad (1)$$

This expression is minimal when $t_l = k'$ for some l , since replacing $t_{l_1}, t_{l_2} > 0$ with $t'_{l_1} = t_{l_1} + t_{l_2}, t'_{l_2} = 0$ strictly decreases its value. Therefore $\Pr[uv \notin E(G^k)] \leq \frac{k'}{\nu} \leq \frac{k}{\nu}$. Notice that, crucially, by the structure of S' , as each edge class appears in at most one pair of vertices of S' , the events $uv \notin E(G^k)$ are independent for different pairs u, v . Let $A_{S'}$ denote the event that there is an independent set S' of the above form of size $s' = \lceil c' \log_2(N) \rceil$, where $c' = 2k^2$. Applying the same consideration used on S and G to S' and G^k , gives (assuming N is sufficiently large):

$$\Pr[A_{S'}] \leq \binom{N^k}{s'} \left(\frac{k}{\nu} \right)^{\binom{s'}{2}} \leq N^{ks'} 2^{-\frac{1}{2}s'^2 \log_2(\frac{\nu}{k})} \leq 2^{(kc' - \frac{1}{2} \log_2(\frac{\nu}{k})c'^2) \log_2^2(N)}$$

Now, our choice of c' should satisfy $c' > \frac{2k}{\log_2(\frac{\nu}{k})}$ for this probability to tend to zero. Whenever $2 \leq k \leq \frac{\nu}{2}$ we get $k \log_2(\frac{\nu}{k}) \geq k > 1$, thus $c' = 2k^2 > \frac{2k}{\log_2(\frac{\nu}{k})}$. For $\frac{\nu}{2} < k < \nu$ we have $1 < \frac{\nu}{k} < 2$ and thus $\log_2(\frac{\nu}{k}) > \frac{\nu}{k} - 1$. Taking any $c' \geq \frac{2k^2}{\nu - k}$ would be sufficient in this case, hence $c' = 2k^2$ will do. Overall, we get that $\Pr[A_{S'}]$ tends to 0 as N tends to infinity.

Altogether, we have shown that for every $2 \leq k < \nu$:

$$\alpha(G^k) \leq k^2 \nu \alpha(G^{k-1}) 2k^2 \log_2(N) = 2k^4 \nu \log_2(N) \alpha(G^{k-1})$$

Hence, plugging in the facts that $\alpha(G) \leq 2 \log_\nu(N) < 2 \log_2(N)$ and $2^{\frac{m}{2}} m! \leq m^m$ for $m \geq 2$, we obtain the following bound for all $k \in \{1, \dots, \nu - 1\}$:

$$\begin{aligned} \alpha(G^k) &\leq 2^k (k!)^4 \nu^{k-1} \log_2^k(N) \leq 2^{-k} k^{4k} \nu^{k-1} \log_2^k(N) \\ a_k &\leq \frac{1}{2} k^4 \nu \log_2(N) \leq \frac{1}{2} \nu^5 \log_2(N) \end{aligned}$$

It remains to show that a_ν is large. Consider the following set of vertices in G^ν (with addition modulo N):

$$I = \{ \bar{x} = (x, x+n, \dots, x+(\nu-1)n) \mid 0 \leq x < N \} \quad (2)$$

Clearly I is independent, since for any $0 \leq x < y < N$, the corresponding coordinates of \bar{x}, \bar{y} form one complete edge class, thus exactly *one* of these coordinates is disconnected in G . This implies that $a_\nu \geq N^{\frac{1}{\nu}}$.

Hence, we have shown that for every value of ν , there exists a graph G on N vertices such that:

$$\begin{cases} a_i \leq c_i \log_2(N) & (i = 1, \dots, \nu - 1) \\ a_\nu \geq N^{\frac{1}{\nu}} \end{cases} \quad (3)$$

■

We note that a simpler construction could have been used, had we wanted slightly weaker results, which are still asymptotically sufficient for proving the theorem. To see this, take $N = n\nu$ and start with the complete graph K_N . Now order the N vertices arbitrarily in n rows (each of length ν), as (v_{ij}) ($1 \leq i \leq n, 1 \leq j \leq \nu$). For each pair of rows i, i' , choose (independently) a single column $1 \leq j \leq \nu$, and remove the edge $v_{ij}v_{i'j}$ from the graph. This gives a graph G with $\binom{N}{2} - \binom{n}{2}\nu$ edges. A calculation similar to the one above shows that with high probability $a_k \leq c_k \log_2(N)$ for $k < \nu$, and yet $\alpha(G^\nu) \geq n$ (as opposed to N in the original construction), hence $a_\nu \geq (\frac{N}{\nu})^{\frac{1}{\nu}} \geq \frac{1}{2} N^{\frac{1}{\nu}}$.

3 Graphs with an irregular independence series

Theorem 2 states that there exists a graph G whose independence series exhibits an arbitrary (finite) number of jumps. Our first step towards proving this theorem is to examine the behavior of fixed powers of the form $k \geq \nu$ for the graphs described in the previous section. We show that these graphs, with high probability, satisfy $a_k = (1 + O(\log N)) N^{\lfloor \frac{k}{\nu} \rfloor \frac{1}{k}}$, for every fixed $k \geq \nu$. The notation $a_k = (1 + O(\log N)) N^\alpha$, here and in what follows, denotes that $N^\alpha \leq a_k \leq cN^\alpha \log N$ for a fixed $c > 0$. The lower bound of $N^{\lfloor \frac{k}{\nu} \rfloor \frac{1}{k}}$ for a_k can be derived from the cartesian product of the set I , defined in (2), with itself, $\lfloor \frac{k}{\nu} \rfloor$ times; the upper bound is more interesting. Fix an arbitrary set S , as before, however, this time, prior to generating S' , we first remove from S all vertices which contain among their coordinates a set of the form $\{x, x+n, \dots, x+(\nu-1)n\}$. This amounts to at most $\binom{k}{\nu} \nu! n \alpha(G^{k-\nu})$ vertices. This step ensures that S will not contain vertices that share a relation, such as the one appearing in the set I defined in (2). However, an edge class may

still be completely contained in the coordinates of $u, v \in S$, in an interlaced form, for instance: $u = (x, y + n, x + 2n, \dots, x + (\nu - 1)n, \dots)$ and $v = (y, x + n, y + 2n, \dots, y + (\nu - 1)n, \dots)$. This will be automatically handled in generating S' , since all vectors v with $x + tn$ in any of their coordinates are removed from S' after processing the vector u . Equation (1) remains valid, with $t_i < \nu$ for all i , however now we must be more careful in minimizing its right hand side. We note that for every $0 < t_i, t_j < \nu - 1$, setting $t'_i = \nu - 1, t'_j = t_i + t_j - t'_i$ reduces the product of $\frac{(\nu - t_i)(\nu - t_j)}{\nu^2}$. Therefore, again denoting by k' the number of distinct pairs of corresponding coordinates, we obtain the following bound on the probability of the edge uv :

$$\Pr[uv \in E(G^k)] \geq \left(\frac{1}{\nu}\right)^{\lfloor \frac{k'}{\nu-1} \rfloor} \frac{\nu - (k' \bmod (\nu - 1))}{\nu} \geq \left(\frac{1}{\nu}\right)^{\frac{k'}{\nu-1}} \geq \left(\frac{1}{\nu}\right)^{\frac{k}{\nu-1}} \quad (4)$$

Thus:

$$\Pr[uv \notin E(G^k)] \leq e^{-(\frac{1}{\nu})^{\frac{k}{\nu-1}}}$$

Now, the same consideration that showed $\alpha(G) \leq 2 \log_\nu(N)$ implies that any set S' generated from S in this manner, which is of size $s' \geq 2k\nu^{\frac{k}{\nu-1}} \log(N)$, is almost surely not independent (for the sake of convenience, we set $p = e^{-(\frac{1}{\nu})^{\frac{k}{\nu-1}}}$). Indeed, the probability that there is such an independent set S' is at most:

$$\binom{N^k}{s'} p^{\binom{s'}{2}} \leq \frac{p^{-s'/2}}{s'!} \left(N^k p^{\frac{s'}{2}}\right)^{s'} = \frac{p^{-s'/2}}{s'!} \exp\left(k \log(N) - \frac{s'}{2} \nu^{-\frac{k}{\nu-1}}\right)^{s'} \leq \frac{p^{-s'/2}}{s'!} = o(1)$$

Thus, almost surely, $|S'| \leq 2k\nu^{\frac{k}{\nu-1}} \log(N)$. Altogether, we have:

$$\begin{aligned} \alpha(G^k) &\leq \binom{k}{\nu} \nu! n \alpha(G^{k-\nu}) + k^2 \nu \alpha(G^{k-1}) \cdot 2k\nu^{\frac{k}{\nu-1}} \log(N) = \\ &= \binom{k}{\nu} (\nu - 1)! N \alpha(G^{k-\nu}) + 2k^3 \nu^{1+\frac{k}{\nu-1}} \log(N) \alpha(G^{k-1}) \end{aligned} \quad (5)$$

For $k = \nu$ and a sufficiently large N , we get

$$N \leq \alpha(G^\nu) \leq N(\nu - 1)! + 2\nu^{5+\frac{1}{\nu-1}} \log(N) (c_{\nu-1} \log_2(N))^{\nu-1} \leq N \log_2(N) \quad (6)$$

Set $d_1 = \dots = d_{\nu-1} = 0$, $d_\nu = 1$ and $d_k = 4k^3 \nu^{1+\frac{k}{\nu-1}} d_{k-1}$ for $k > \nu$. It is easy to verify that $\frac{1}{2}d_k \geq \binom{k}{\nu} (\nu - 1)! d_{k-\nu}$, and $\frac{1}{2}d_k \geq 2k^3 \nu^{1+\frac{k}{\nu-1}} d_{k-1}$. Hence, by induction, equations (5) and (6) imply that for all $k \geq \nu$:

$$\alpha(G^k) \leq d_k N^{\lfloor \frac{k}{\nu} \rfloor} \log_2^k(N)$$

By definition of the d_k series,

$$d_k \leq 4^{k-\nu} \left(\frac{k!}{\nu!}\right)^3 \nu^{(k-\nu)(1+\frac{k}{\nu-1})} \leq 4^k (k!)^3 \nu^{k(1+\frac{k-\nu}{\nu-1})} = 4^k (k!)^3 \nu^{\frac{k-1}{\nu-1}}$$

Hence,

$$1 \leq \frac{a_k}{N^{\lfloor \frac{k}{\nu} \rfloor \frac{1}{k}}} \leq \sqrt{2} k^3 \nu^{\frac{k-1}{\nu-1}} \log_2(N)$$

as required.

Let us construct a graph whose independence series exhibits two jumps (an easy generalization will provide any finite number of jumps). Take a random graph, G_1 , as described above, for some index ν_1 and a sufficiently large number of vertices N_1 , and another (independent) random graph, G_2 for some other index $\nu_2 > \nu_1$, on $N_2 = N_1^{\alpha \frac{\nu_2}{\nu_1}}$ vertices (when $\alpha > 1$). Let $G = G_1 \cdot G_2$ be the strong product of the two graphs; note that G has $N = N_1 N_2$ vertices. It is crucial that we do not take G_1 and G_2 with jumps at indices ν_1, ν_2 respectively *separately*, but instead consider the product G of two random graphs constructed as above. We claim that with high probability, G satisfies:

$$a_k(G) = \begin{cases} O(\log N) & k < \nu_1 \\ (1 + O(\log N)) N_1^{\lfloor \frac{k}{\nu_1} \rfloor \frac{1}{k}} & \nu_1 \leq k < \nu_2 \\ (1 + O(\log N)) N_2^{\lfloor \frac{k}{\nu_2} \rfloor \frac{1}{k}} & k \geq \nu_2 \end{cases}$$

i.e., for $\nu_1 \leq k < \nu_2$ which is a multiple of ν_1 , we have $a_k = (1 + O(\log N)) N^{\frac{1}{\nu_1 + \alpha \nu_2}}$; for $k \geq \nu_2$ which is a multiple of ν_2 we have $a_k = (1 + O(\log N)) N^{\frac{\alpha}{\nu_1 + \alpha \nu_2}}$. Therefore, we get an exponential increase of order α at the index ν_2 , and obtain two jumps, as required.

To prove the claim, argue as follows: following the formerly described methods, we filter an arbitrary set $S \subset V(G^k)$ to a subset S' , in which every two vertices have a positive probability of being connected, and all such events are independent. This filtering is done as before - only now, we consider the criteria of both G_1 and G_2 when we discard vertices. In other words, if we denote by u^1, u^2 the k -tuples corresponding to G_1^k and G_2^k of a vertex $u \in V(G^k)$, then a vertex $u \in S$ filters out the vertex v from S iff u^1 would filter out v^1 in G_1^k or u^2 would filter out v^2 in G_2^k (or both). Recall that, by the method S' is generated from S , no two vertices in S' share an identical k -tuple of G_1^k or of G_2^k . Hence, two vertices $u, v \in S'$ are adjacent in G^k iff they are adjacent both in G_1^k and in G_2^k . These are two independent events, thus, by (1) and (4), we get the following fixed lower bound on the probability of u and v being adjacent:

$$\Pr[uv \in E(G^k)] = \Pr[u^1 v^1 \in E(G_1^k)] \Pr[u^2 v^2 \in E(G_2^k)] \geq \Omega(1)$$

This provides a bound of $O(\log N)$ for the size of S' . Combining this with the increase in the values of $\{a_k\}$ at indices ν_1 and ν_2 ($a_{\nu_i} \geq N_i^{\frac{1}{\nu_i}}$ for $i = 1, 2$) proves our claim.

In order to obtain any finite number of jumps, at indices ν_1, \dots, ν_s , simply take a sufficiently large N_1 and set $N_i = N_{i-1}^{\alpha \frac{\nu_i}{\nu_{i-1}}}$ for $1 < i \leq s$, where $\alpha > 1$. By the same considerations used above, with high probability the graph $G = G_1 \cdot \dots \cdot G_s$ (where G_i is a random graph designed to have a jump at index ν_i almost surely) satisfies $a_{\nu_i} \geq a_k^\alpha$ for all $k < \nu_i$. Hence for every $\varepsilon > 0$ we can choose $\alpha > \frac{1}{\varepsilon}$ and a sufficiently large N_1 so that $a_k < a_{\nu_i}^\varepsilon$ for all $k < \nu_i$. This completes the proof. ■

4 Concluding remarks and open problems

We have shown that even when the independence series stabilizes for an arbitrary (fixed) number of elements, or jumps and then stabilizes, it still does not necessarily approximate the Shannon capacity up to any power of $\varepsilon > 0$. However, our constructions require the number of vertices to be exponentially large in the values of the jump indices ν_i . We believe that this is not a coincidence, namely a prefix of *linear* (in the number of vertices) length of the independence series can provide a good approximation of the Shannon capacity. The following two specific conjectures seem plausible:

Conjecture 3. *For every graph G on n vertices, $\max\{a_k\}_{k \leq n} \geq \frac{1}{2}c(G)$, that is, the largest of the first n elements of the independence series gives a 2-approximation for $c(G)$.*

Conjecture 4. *For every $\varepsilon > 0$ there exists an $r = r(\varepsilon)$ such that for a sufficiently large n and for every graph G on n vertices, the following is true: $\max\{a_k\}_{k \leq n^r} \geq (1 - \varepsilon)c(G)$.*

Our proof of Theorem 1 shows the existence of a graph whose independence series increases by a factor of N^δ at the k -th power, where $\delta = \frac{1-o(1)}{k}$. It would be interesting to decide if there is a graph satisfying this property for a *constant* $\delta > 0$ (independent of k). This relates to a question on channel discrepancy raised in [3], where the authors show that the ratio between the independence number and the Shannon capacity of a graph on n vertices can be at least $n^{\frac{1}{2}-o(1)}$, and ask whether this is the largest ratio possible. Proving Theorem 1 for a constant $\delta > 0$ will give a negative answer for the following question, which generalizes the channel discrepancy question mentioned above:

Question 5. *Does $\max\{a_i\}_{i \leq k}$, for any fixed $k \geq 2$, approximate $c(G)$ up to a factor of $n^{\frac{1}{k}+o(1)}$ (where $n = |V(G)|$)?*

Although our results exhibit the difficulty in approximating the Shannon capacity of a given graph G , this problem is not even known to be NP-hard (although it seems plausible that it is in fact much harder). We conclude with a question concerning the complexity of determining the value of $c(G)$ accurately for a given graph G :

Question 6. *Is the problem of deciding whether the Shannon Capacity of a given graph exceeds a given value decidable?*

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